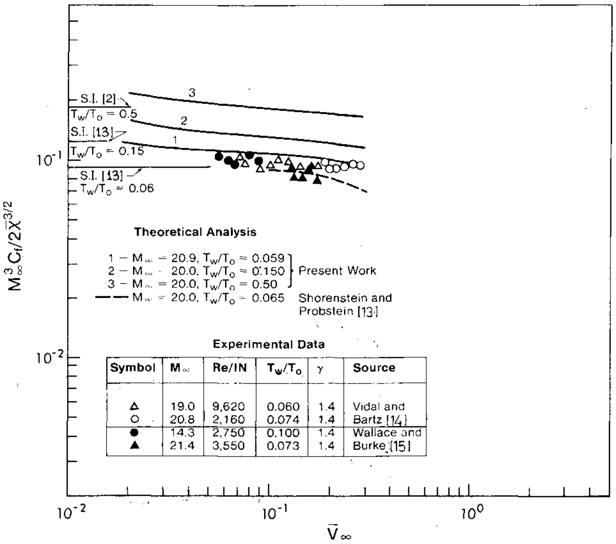

Fig. 2 Variation of skin friction on a sharp flat plate as a function of rarefaction parameter (cooled surface).

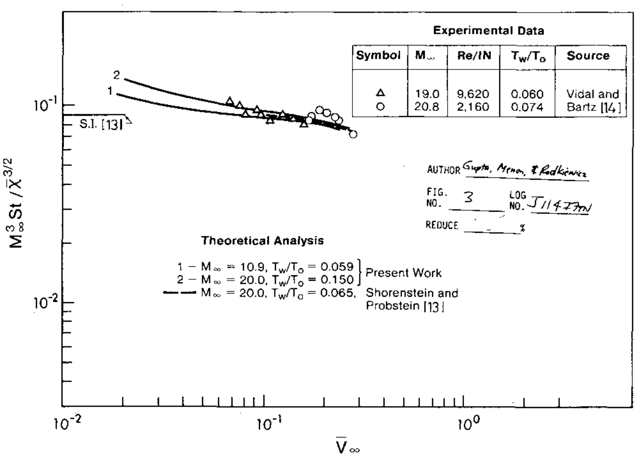

Fig. 3 Variation of Stanton number on a sharp flat plate as a function of rarefaction parameter (cooled surface).

measurements[14] is quite good, as shown in Fig. 3. As before, Shorenstein and Probstein's analysis[13] tends towards the present results in the downstream limit. It may be seen from Figs. 2 and 3 that as the plate is progressively cooled, both skin friction and heat transfer decrease because of the lower viscosity and thermal conductivity values at the cooled surface.

Finally, the present analysis does give a picture of the flowfield in the region downstream of the viscous shock layer that is qualitatively correct (i.e., the tendency of the experimental data to level off at large values of $\bar{\chi}$ is represented) and numerically accurate within the framework of the boundary layer theory.

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Boundary Layer of Density-Stratified Fluids with a Suspension of Particles

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Introduction

THE study of the fluid dynamics of a particulate suspension (where the suspended matter may consist of solid particles, liquid droplets, gas bubbles, etc.) is of interest in a wide range of areas of scientific and technical importance. The theoretical study of this system of fluids has been very useful in understanding such phenomena as sedimentation, fluidization, combustion, atmospheric fallout, electrostatic precipitation of dust, nuclear reactor cooling, flows in rocket tubes, lunar ash flows, environmental pollution, aerosol and paint spraying, aircraft icing, and, more recently, blood flow. The study of boundary layers of the flow of particulate suspensions is important in collecting much useful information, including information concerning particle accumulation, retardation, and impingement on solid surfaces. These studies include the work of Marble,[1] Chiu,[2] Singleton,[3,4] Soo,[5] and Zung.[6] Tabakoff and Hamed[7] analyzed boundary layers of particulate flow in cascades in detail using the momentum integral method. They found that the presence of particles leads to an increase in the gas

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boundary layer thickness. Later, Chakrabarti[8] discussed boundary-layer theory for fluid-particle flow past an infinite plate in the initial stages, using a perturbation method. The purpose of this Note is to study boundary-layer theory of viscous incompressible fluids in which identical rigid spherical particles are distributed nonuniformly. Very few studies of such fluids have been reported so far. These studies include the work of Kelly and Redekopp,[9] Standing,[10] Martin,[11] and Pao.[12] Besides other areas, the result might also be applicable to blood flow where the spatial distribution of red blood cells is nonuniform [lower number density of red blood cells close to the wall than in the center (McDonald[13])].

Formulation of the Problem

Let the velocities of the fluid phase be specified by $u(y,t)$ and those of the particulate phase by $v(y,t)$. Suppose $\rho(y)$ and $\rho_p(y)$ denote the densities of the two phases, respectively. It is assumed that the motion is induced by the potential flow $u_\infty(x,t)=U(x)\cos\omega t$, where ω is the frequency. The appropriate equations given by Ref. 8 are

$$\frac{\partial u}{\partial t}=\frac{\mu\tau}{\rho(y)}\frac{\partial^2 u}{\partial y^2}-f(u-v) \tag{1}$$

$$\frac{\partial v}{\partial t}=u-v \tag{2}$$

where μ (constant) is the viscosity of the fluid; t is the time divided by τ, the relaxation time of the particle; $f=\rho_p/\rho$. The boundary conditions are $y=0$, $u=0$; $y=\infty$, $u=U(x)\cos\omega t$. The initial conditions are $t=0$, $u=U(x)$, $v=0$.

Solution of the Problem

Let us assume the variation of ρ with y as $\rho=\rho_0 e^{-cy}$, where ρ_0 is the density at $y=0$. It is further assumed that the particle density is also decreasing exponentially with y, as a result of which $f=\rho_p/\rho$, the mass concentration of the particle, can be taken as a constant. We will express u and v as $u=u_0+fu_1+f^2u_2+\ldots, v=v_0+fv_1+f^2v_2+\ldots$. Here $u=u_0$, $v=v_0$ are taken as first approximations and $u=u_0+fu_1$, $v=v_0+fv_1$ are taken as second approximations.

Case 1: Oscillatory Motion

A Laplace transformation and $r^2=\exp(-cy)$ are now used to obtain the solution for fluid and particle velocities (first approximation) and are given as

$$u_0(y,t)=U(x)\,(U_y^2+V_y^2)^{1/2}\cos(\omega t-\gamma) \tag{3}$$

$$v_0(y,t)=U(x)\,[U_y^2+V_y^2]^{1/2}/(1+\omega^2)\,[\cos(\omega t-\gamma)+\omega\sin(\omega t-\gamma)+e^{-t}(\omega\sin\gamma-\cos\gamma)] \tag{4}$$

Here U_y and V_y are and imaginary parts of $I_0(Re^{i\theta})$, $R=\alpha e^{-cy/2}$, $\theta=\pi/4$, $\gamma=\tan^{-1}(-V_y/U_y)$, $\alpha=\sqrt{\omega/\nu'}$, and $\nu'=\mu\tau c^2/4\rho_0$.

The functions U and V have been tabulated and graphed by Briggs and Lowan.[14] These two relations above may also be written as

$$u_0(y,t)=U(x)\,[\cos\omega t\,\mathrm{ber}(\alpha e^{-cy/2})-\sin\omega t\,\mathrm{bei}(\alpha e^{-cy/2})] \tag{5}$$

$$v_0(y,t)=U(x)/(1+\omega^2)\,[(\cos\omega t-e^{-t})A(y)+\sin\omega t B(y)] \tag{6}$$

where $A(y)=\mathrm{ber}(\alpha e^{-cy/2})+\omega\,\mathrm{bei}(\alpha e^{-cy/2})$, $B(y)=\omega\,\mathrm{ber}(\alpha e^{-cy/2})-\mathrm{bei}(\alpha e^{cy/2})$, and "ber" and "bei" are real and imaginary parts of $I_0(ze^{\pi i/4})$. The solution in the case of a second approximation is carried out by using standard techniques and (to a second approximation) yields the following steady oscillatory solution for the fluid and particle velocities:

$$u(y,t)=U(x)\,[(\cos\omega t\,\mathrm{ber}(\alpha e^{-cy/2})-\sin\omega t\,\mathrm{bei}(\alpha e^{-cy/2})+f/(1+\omega^2)g(y,t)] \tag{7}$$

$$v(y,t)=\frac{U(x)}{1+\omega^2}[\{(\cos\omega t-e^{-t})A(y)+\sin\omega t B(y)\}+fe^{-t}\int_0^t g(y,t)e^t\mathrm{d}t] \tag{8}$$

where

$$g(r,t)=\int_0^1 F(\eta,r,t)\mathrm{d}\eta,\quad F(\eta,r,t)=(L+M)/\nu'$$

$$L=\pi/2e^{-t}[\mathrm{ber}(\alpha\eta)-\omega\,\mathrm{bei}(\alpha\eta)][Y_0(\beta r)J_0(\beta)-Y_0(\beta)J_0(\beta r)]J_0(\beta\eta)/J_0(\beta)$$

$$M=(i+\omega)e^{i\omega t}/2(1+\omega^2)\{(1+\omega^2)\mathrm{ber}(\alpha\eta)+\omega(1+i)(1+i\omega)\mathrm{bei}(\alpha\eta)\}\times[K_0(r)I_0(\gamma' r)-I_0(\gamma')K_0(\gamma' r)]I_0(\gamma'\eta)/I_0(\gamma')+\text{conjugate}$$

where $\beta=\sqrt{1/\nu'}$, $\gamma'=\sqrt{i\omega/\nu'}$, and $K_0(z)$, $Y_0(z)$ are modified Bessel functions.

Steady Oscillatory Skin Friction

Along with the fluid and particulate phase velocities, the steady oscillatory skin friction is also of interest. Thus, if P_{fc} denotes the steady oscillatory skin friction, one obtains

$$P_{fc}=-\mu\alpha cU(x)/2[\cos\omega t\,\mathrm{ber}'(\alpha)-\sin\omega t\,\mathrm{bei}'(\alpha)+f/(1+\omega^2)g'(0,t)] \tag{9}$$

where primes denote differentiation with respect to y and the subscript c denotes stratification. The dimensionless skin friction ratio is then

$$P_{fc}/P_{0c}=1+fg'(0,t)/(1+\omega^2)[\cos\omega t\,\mathrm{ber}'(\alpha)-\sin\omega t\,\mathrm{bei}'(\alpha)] \tag{10}$$

Case 2: Impulsive Motion

By consecutive use of Hankel transformation with $r^2=\exp(-cy)$ and the boundary condition $u_0(\infty,t)=U(x)$, one finds expressions for fluid and particle velocities (to a second approximation) as

$$u(y,t)=2U(x)\sum_{i=1}^{\infty}\left\{e^{-\nu'\xi_i^2 t}+\frac{f[e^{-t}-(\nu'\xi_i^2)^2e^{-\nu'\xi_i^2 t}]}{(1-\nu'\xi_i^2)}\right\}\times\frac{J_0(\xi_i e^{-cy/2})}{\xi_i J_1(\xi_i)} \tag{11}$$

$$v(y,t)=2U(x)\sum_{i=1}^{\infty}\left[e^{-\nu'\xi_i^2 t}-e^{-t}+f\left\{te^{-t}-\frac{(\nu'\xi_i^2)^2}{1-\nu'\xi_i^2}\times(e^{-\nu'\xi_i^2 t}-e^{-t})\right\}\right]J_0(\xi_i e^{-cy/2})/(1-\nu'\xi_i^2)\xi_i J_1(\xi_i) \tag{12}$$

where the ξ_i $(i=1,2,3\ldots)$ are roots of $J_0(\xi)=0$.

Local Skin Friction

If $Q_{fc}(x)$ is the local skin friction for the fluid with a suspension of particles, then

$$Q_{fc}(x)=c\mu U(x)\sum_{i=1}^{\infty}[e^{-\nu'\xi_i^2 t}+f\{e^{-t}-(\nu'\xi_i^2)^2e^{-\nu'\xi_i^2 t}\}/(1-\nu'\xi_i^2)] \tag{13}$$

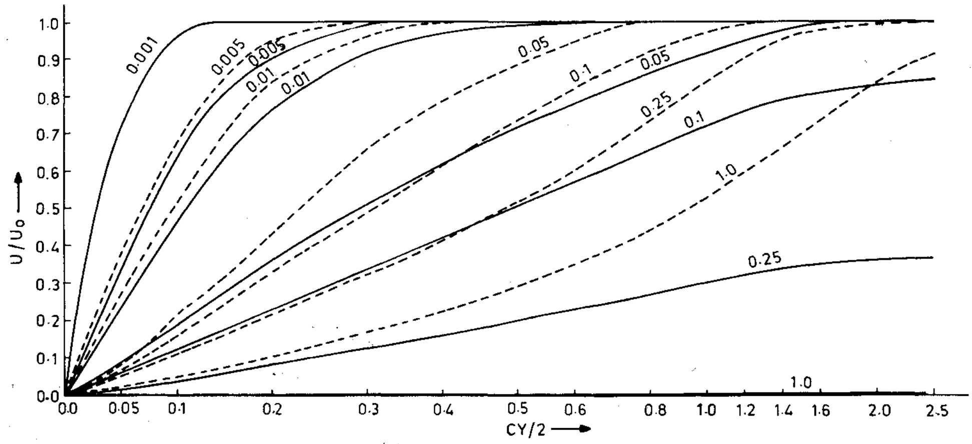

Fig. 1 Fluid velocity profiles for $f=0$ up to first approximation; numbers $\nu' t$. (——), stratified flow, (– – –) nonstratified flow.[8]

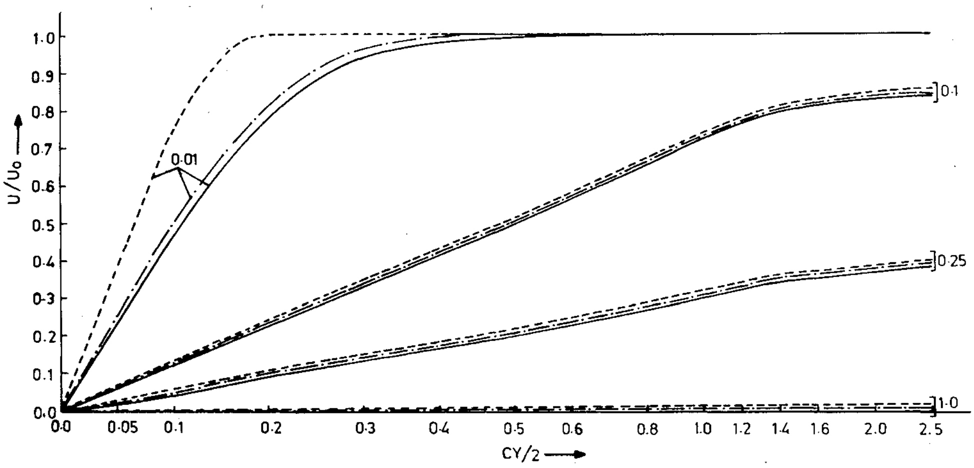

Fig. 2 Fluid velocity profiles for $\nu' = 0.01$, numbers $\nu' t$. (——), $f=0$; (– · –), $f=0.1$; (· · · ·), $f=1$.

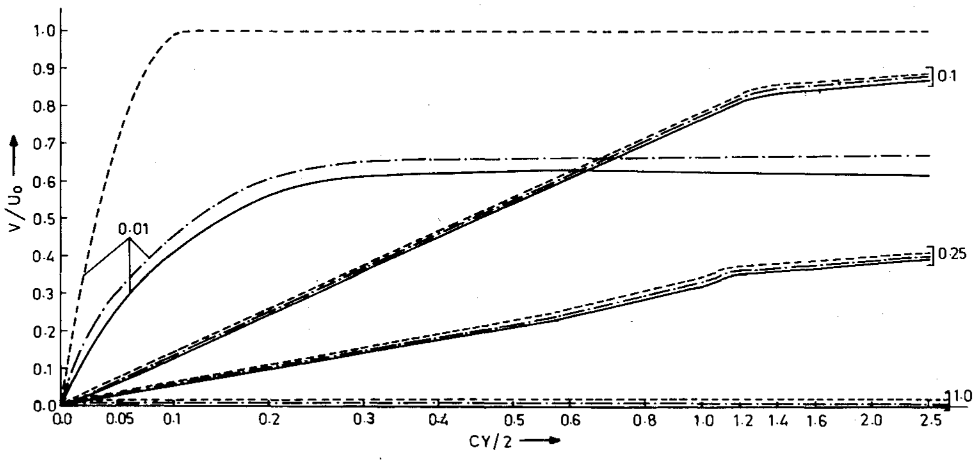

Fig. 3 Particle velocity profiles for $\nu' = 0.01$, numbers $\nu' t$ (——), $f=0$; (– · –), $f=0.1$; (· · · ·), $f=1$.

Thus, the dimensionless local skin friction ratio is given by

$$Q_{fc}(x)/Q_{0c}(x) = 1 + f\sum_{i=1}^{\infty}\left[(e^{-t} - (\nu'\xi_i^2)^2 e^{-\nu'\xi_i^2 t})/(1-\nu'\xi_i^2)\right]\Big/\sum_{i=1}^{\infty} e^{-\nu'\xi_i^2 t} \tag{14}$$

Other Approximate Numerical Results and Discussions

Large time solution. For large time, only $\Sigma\xi_i = \xi_1$ is significant. In this case, when ν' is not very large, the velocity of the fluid from Eq. (11) is

$$u \sim 2U(x) J_0(\xi_1 e^{-cy/2})/\xi_1 J_1(\xi_1) e^{-\nu'\xi_1^2 t} \tag{15}$$

Thus the two-phase system behaves as a single fluid; that is the equilibrium state has been reached. The large time solution for particle-free flow from Eq. (11) is found to be the same as that from Eq. (15). Thus it may be concluded here that for large time with small density ratio f, suspension flow approaches particle-free flow.

The local skin friction ratio for large time with small density ratio is, from Eq. (15),

$$Q_{fc}/Q_{0c} \sim 1 \tag{16}$$

For particle-free flow, the ratio of the local skin friction in the stratified flow to the local skin friction in the flow with

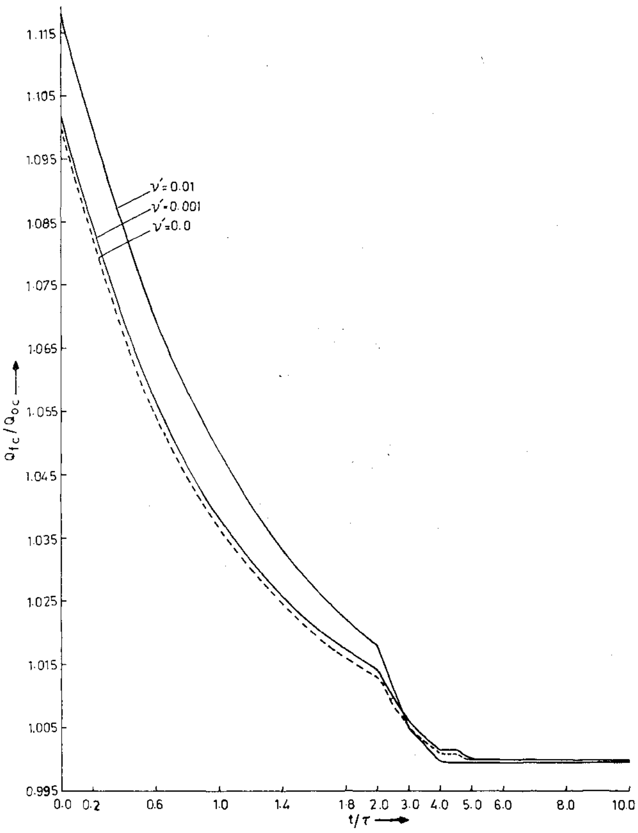

Fig. 4 Wall shear stress, $f=0.1$. (—), stratified flow, (---) nonstratified flow.

constant density is, from Eq. (14),

$$Q_{0c}/Q_{00} \sim c\mu U(x) \sum_{i=1}^{\infty} e^{-\nu' \xi_i^2 t} / \mu U(x) (\pi \nu t)^{1/2}$$

$$\sim 2(\pi \nu' t)^{1/2} \sum_{i=1}^{\infty} e^{-\nu' \xi_i^2 t} \tag{17}$$

For large time this ratio is

$$Q_{0c}/Q_{00} \sim 2(\pi \nu' t)^{1/2} e^{-\nu' \xi_i^2 t} \tag{18}$$

Particle velocity as $y \to \infty$ with $\omega = 0$ is, from Eq. (4),

$$v_0 \sim U(x)(1-e^{-t}) \tag{19}$$

which is the same as Eq. (16) of Chakrabarti's paper (Ref. 8).

Equation (19) shows that the difference between particle velocity and potential velocity $U(x)$ of the fluid in the mainstream decreases with increase in time and for the large time system attains equilibrium when $v_0 \to U(x)$ as $t \to \infty$.

To gain insight into the patterns of flow, the velocities of fluid containing particles, particles, and fluid free from particles have been plotted (Figs. 1, 2, and 3) for both density-stratified and nonstratified cases. Figures 1 and 2 represent the fluid velocity profiles. Since $\mathrm{erf}(y/2\sqrt{\mu/\rho_0 t}) = \mathrm{erf}(cy/2/2\sqrt{\nu' t})$, the velocity profiles of a density-nonstratified fluid for the first approximation of Chakrabarti's paper (Ref. 8) are also shown in Fig. 1, which reveals that when time is small, say $\nu' t < 0.005$, the stratification has a very small effect on the flow. But, when $\nu' t > 0.005$, the difference between the stratified and nonstratified profiles increases rapidly with increase in time, in particular at large distance from the plate. Wall stress ratios for the suspension have been plotted in Figs. 4 and 5. When $\nu' \leq 0.001$, the stratification has a very small effect on the stress ratio (Fig. 4). Figures 4 and 5 further reveal that for any given f, the stress ratio rapidly approaches the nonstratified values as $\nu' \to 0$. Also,

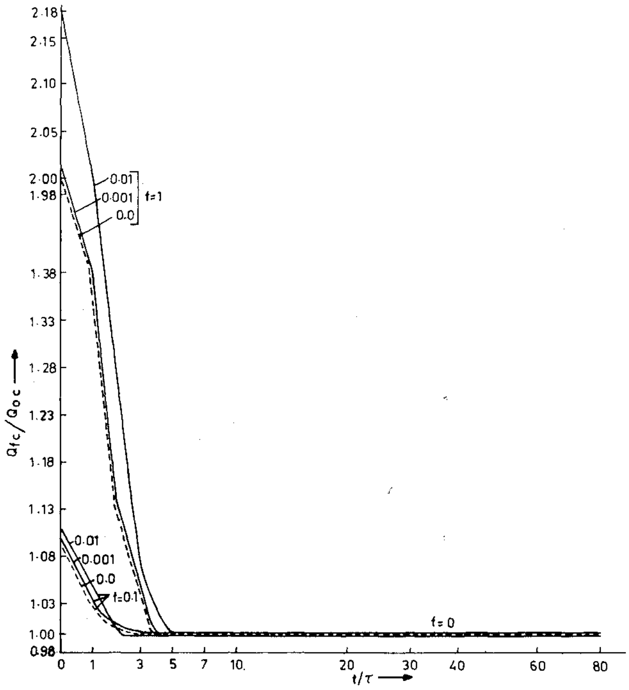

Fig. 5 Wall shear stress, numbers ν'.

the shear stress ratio approaches 1 as time increases for any given value of f and ν'.

Acknowledgments

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